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Operator for describing polarization states of a photon in terms of Riemann-Silberstein quantized electromagnetic vector*

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Abstract. Based on the quantized electromagnetic field described by the Riemann-Silberstein complex vector **F**, we construct the eigenvector set of **F**, which makes up an orthonormal and complete representation. In terms of **F** we then introduce a new operator which can describe the relative ratio of the left-handed and right-handed polarization states of a polarized photon. In **F**′s eigenvector basis the operator manifestly exhibits a behaviour which is similar to a phase difference between two orientations of polarization of a light beam in classical optics.

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1 Introduction

In recent years, single photon states has attracted much attention of physicist because they exhibit high powerlaw falloff of the energy density and of the photon detection rates [1–3]. The photon states are conveniently described in terms of the Riemann-Silberstein complex vector of electromagnetic field [4]. The spatial localization properties of one photon states and the decay in time has been analyzed in reference [3]. In the present letter we seek a new quantum mechanical formalism to determine the nature of the polarization of a single photon, which is based on the quantized Riemann-Silberstein complex vector.

As for the polarization of single photon states let us quote the idea of Dirac [5]: "When we make the photon meet a tourmaline crystal, we are subjecting it to an observation. We are observing whether it is polarized parallel or perpendicular to the optic axis. The effect of making this observation is to force the photon entirely into the state of parallel or entirely into the state of perpendicular polarization. It has to make a sudden jump from being partly in each of these two states to being entirely in one or other of them. Which of the two states it will jump into cannot be predicted, but is governed only by probability laws. If it jumps into the parallel state it gets absorbed and if it jumps into the perpendicular state it passes through the crystal and appears on the other side preserving this state of polarization". In this work we want to introduce a new operator which can describe the relative ratio of the two polarizations (left-handed and right-handed) for a photon polarized obliquely to the optic axis. Moreover, when we establish a new representation $|\xi\rangle_k$ for the quantized Riemann-Silberstein complex vector of electromagnetic field, it turns out that in $|\xi\rangle_k$ representation the new operator can manifestly exhibit a behaviour which is similar to a phase difference between two orientations of polarization for a beam in classical optics. Recall that for a beam of classical electromagnetic plane wave the electric fields $E_x = A_1\cos(\tau+\delta_1)$, $E_y = A_2\cos(\tau+\delta_2)$ can be re-written as (eliminating τ between these two equations)

$$\left(\frac{E_x}{A_1}\right)^2 + \left(\frac{E_y}{A_2}\right)^2 - 2\frac{E_x}{A_1}\frac{E_y}{A_2}\cos\delta = \sin^2\delta,\qquad \delta = \delta_2 - \delta_1, \quad (1)$$

where $\exp(\mathrm{i}\delta)$ is the phase difference as a parameter describing the polarization [6]. For example, for a right-handed polarized electric wave, $E_y/E_x = \mathrm{e}^{\mathrm{i}\delta} = \mathrm{e}^{-\mathrm{i}\pi/2}$. The work is arranged as follows: in Section 2 we construct the eigenvector of the quantized Riemann-Silberstein complex field. In Section 3 we introduce the operator for the description of photon polarization states in terms of the quantized Riemann-Silberstein complex vector. In Section 4 we point out that although our new operator has nothing to do with optical phase in the usual sense which was discussed in [7–9] by Dirac, Susskind-Glogower and Lynch, we compare and contrast the operator with Noh,

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Fougeres and Mandel (NFM) operationally defined phase operator [10,11] because they resemble to each other in form.

2 Eigenvector of quantized Riemann-Silberstein complex vector

As shown in [1], the most general one-photon (1ph) state can be described by two complex functions of the wave vector

$$|1\mathrm{ph}\rangle = \int \mathrm{d}^3k f_+(\mathbf{k}) a^\dagger(\mathbf{k})|00\rangle_k + \int \mathrm{d}^3k f_-(\mathbf{k}) b^\dagger(\mathbf{k})|00\rangle_k, \tag{2}$$

where $f_\pm$ are two components of the photon wave function in momentum representation which is normalized to one

$$\int \mathrm{d}^3k |f_+(\mathbf{k})|^2 + \int \mathrm{d}^3k |f_-(\mathbf{k})|^2 = 1 \tag{3}$$

and $a^\dagger(\mathbf{k})$, $b^\dagger(\mathbf{k})$ are the creation operators of photons with the left-handed and right-handed polarization, satisfying

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = \delta(\mathbf{k}-\mathbf{k}') = [b(\mathbf{k}), b^\dagger(\mathbf{k}')] \ . \tag{4}$$

Equations (2–4) coincide with Dirac's idea [5] that: "It is supposed that a photon polarized obliquely to the optic axis may be regarded as being partly in the state of polarization perpendicular to the axis. The state of oblique polarization may be considered as the result of some kind of superposition process applies to the two states of parallel and perpendicular polarization. This implies a certain special kind of relationship between the various states of polarization, a relationship similar to that between polarized beams in classical optics, but which is now to be applied, not to beams, *but to the states of polarization of one particular photon*. This relationship allows any state of polarization to be resolved into, or expressed as a superposition of, any two mutually perpendicular states of polarization".

The photon states are conveniently described in terms of the Riemann-Silberstein complex vector $\mathbf{F}$ (RS vector) composed of electric displacement and the magnetic induction vectors [4],

$$\mathbf{F}(\mathbf{r},t) = \frac{\mathbf{D}(\mathbf{r},t)}{\sqrt{2\epsilon_0}} + \mathrm{i}\frac{\mathbf{B}(\mathbf{r},t)}{\sqrt{2\mu_0}} \cdot \tag{5}$$

The square roots of ϵ and μ are needed to match the dimensions of the two terms and an additional factor of $\sqrt{2}$ is introduced to make the modulus of $\mathbf{F}$ equal simply to the energy density

$$H_{\mathrm{CL}}(\mathbf{r},t) = \frac{\mathbf{D}^2(\mathbf{r},t)}{2\epsilon_0} + \frac{\mathbf{B}^2(\mathbf{r},t)}{2\mu_0} = \mathbf{F}^\dagger(\mathbf{r},t)\cdot\mathbf{F}(\mathbf{r},t) \ . \tag{6}$$

After quantizing the electromagnetic field, the RS vector becomes the field operator $\hat{\mathbf{F}}(\mathbf{r},0)$ (we consider $t=0$ case in Schrödinger picture)

$$\hat{F}(\mathbf{r},0) = \int \mathrm{d}^3k \sqrt{\frac{\hbar c\mathbf{k}}{(2\pi)^3}} e(\mathbf{k}) F(\mathbf{k},\mathbf{r}) \tag{7}$$

where $e(\mathbf{k})$ is an unit polarization vector, and

$$F(\mathbf{k},\mathbf{r}) = a(\mathbf{k}) f_k + b^\dagger(\mathbf{k}) f_k^*, \quad F^\dagger(\mathbf{k},\mathbf{r}) = a^\dagger(\mathbf{k}) f_k^* + b(\mathbf{k}) f_k \tag{8}$$

here $f_k = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}$. We now introduce the eigenvectors of $F(\mathbf{k},\mathbf{r})$:

$$|\xi\rangle_k = \exp\Big(-\frac{|\xi|^2}{2} + \xi f_k^{-1} a^\dagger(\mathbf{k}) + \xi^* f_k^{-1} b^\dagger(\mathbf{k}) - a^\dagger(\mathbf{k}) b^\dagger(\mathbf{k}) f_k^{-2}\Big)|00\rangle_k \tag{9}$$

where $\xi = \xi_1 + \mathrm{i}\xi_2$ is a complex number and the vacuum state $|00\rangle_k$ is annihilated by both $a(\mathbf{k})$ and $b(\mathbf{k})$. We prove here that it is an eigenstate of the operator $F(\mathbf{k},\mathbf{r})$ by acting $a(\mathbf{k})$ on $|\xi\rangle_k$

$$a(\mathbf{k})|\xi\rangle_k = \Big[a(\mathbf{k}), \exp\Big(-\frac{|\xi|^2}{2} + \xi f_k^{-1} a^\dagger(\mathbf{k}) + \xi^* f_k^{-1} b^\dagger(\mathbf{k}) - a^\dagger(\mathbf{k}) b^\dagger(\mathbf{k}) f_k^{-2}\Big)\Big]|00\rangle_k = (\xi f_k^{-1} - b^\dagger(\mathbf{k}) f_k^{-2})|\xi\rangle_k. \tag{10}$$

It then follows

$$F(\mathbf{k},\mathbf{r})|\xi\rangle_k \equiv (a(\mathbf{k}) f_k + b^\dagger(\mathbf{k}) f_k^{-1})|\xi\rangle_k = \xi|\xi\rangle_k. \tag{11}$$

On the other hand, by acting $b(\mathbf{k})$ on $|\xi\rangle_k$, we have

$$b(\mathbf{k})|\xi\rangle_k = (\xi^* f_k^{-1} - a^\dagger(\mathbf{k}) f_k^{-2})|\xi\rangle_k \tag{12}$$

which yields

$$F^\dagger(\mathbf{k},\mathbf{r})|\xi\rangle_k \equiv (b(\mathbf{k}) f_k + a^\dagger(\mathbf{k}) f_k^{-1})|\xi\rangle_k = \xi^*|\xi\rangle_k. \tag{13}$$

Thus $|\xi\rangle_k$ is the common eigenvector of F and $F^\dagger$, which agree with the commutator

$$[F(\mathbf{k},\mathbf{r}), F^\dagger(\mathbf{k},\mathbf{r})] = 0.$$

Using the technique of integration within an ordered product (IWOP) of operators [12] and

$$|00\rangle_{k\cdot k}\langle 00| =: \exp(-a^\dagger(\mathbf{k})a(\mathbf{k}) - b^\dagger(\mathbf{k})b(\mathbf{k})) :$$

(where : : denotes normal ordering) we can perform the following integration (in the following for brevity we write $a(\mathbf{k})$ as a, f_k as f and so on)

$$\int \frac{\mathrm{d}^2\xi}{\pi}|\xi\rangle_{k\cdot k}\langle\xi| = \int \frac{d^2\xi}{\pi} : \exp[-|\xi|^2 + \xi(f^* a^\dagger + fb) + \xi^*(fa + f^* b^\dagger) - a^\dagger b^\dagger f^{-2} - abf^2 - a^\dagger a - b^\dagger b] := \mathbf{1}. \tag{14}$$

This indicates that $|\xi\rangle_k$ make up a completeness relation. Further, by examining equations (11–13) we see

$$\begin{aligned} {}_k\langle\xi'|F(\mathbf{k},\mathbf{r})|\xi\rangle_k &= \xi\ {}_k\langle\xi'|\xi\rangle_k = \xi'\ {}_k\langle\xi'|\xi\rangle_k; \\ {}_k\langle\xi'|F^\dagger(\mathbf{k},\mathbf{r})|\xi\rangle_k &= \xi^*\ {}_k\langle\xi'|\xi\rangle_k = \xi'^*\ {}_k\langle\xi'|\xi\rangle_k, \end{aligned} \tag{15}$$

thus $(\xi'-\xi)\ {}_k\langle\xi'|\xi\rangle_k = (\xi'^*-\xi^*)\ {}_k\langle\xi'|\xi\rangle_k = 0$ which indicates the orthonormal property

$${}_k\langle\xi'|\xi\rangle_k = \pi\delta(\xi'-\xi)\delta(\xi'^*-\xi^*). \tag{16}$$

3 Operator for describing photon polarization states

For describing polarization states of single photon we now introduce a new operator

$$\mathrm{e}^{\mathrm{i}\hat{\Theta}} \equiv \sqrt{\frac{F}{F^\dagger}} = \sqrt{\frac{af+b^\dagger f^*}{a^\dagger f^*+bf}}\,. \tag{17}$$

This definition is feasible as $[af+b^\dagger f^*,\ a^\dagger f^*+bf]=0$ and they can reside within the same square root without ambiguity. Physically, since $a^\dagger(\mathbf{k})$, $b^\dagger(\mathbf{k})$ are creation operators of left-handed and right-handed polarization respectively, $(af+b^\dagger f^*)/(a^\dagger f^*+bf)$ represents the relative ratio of the two polarizations. This phase-difference between two polarizations which is similar to $\mathrm{e}^{\mathrm{i}\delta}$ in classical optics (see Eq. (1)) can be seen more clearly in our $|\xi\rangle_k$ bases. By using equations (11, 13) we have

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\hat{\Theta}} &= \sqrt{\frac{F}{F^\dagger}} = \int \frac{\mathrm{d}^2\xi}{\pi}\sqrt{\frac{\xi}{\xi^*}}|\xi\rangle_{k\cdot k}\langle\xi| \\ &= \int \frac{\mathrm{d}^2\xi}{\pi}\mathrm{e}^{\mathrm{i}\theta}|\xi\rangle_{k\cdot k}\langle\xi| \end{aligned} \tag{18}$$

where $\xi = |\xi|\mathrm{e}^{\mathrm{i}\theta}$. $\mathrm{e}^{\mathrm{i}\theta}$ provides an involved description of the state of polarization of the field. One can immediately determine from the value of this ratio the nature of the polarization. Writing $\mathrm{d}^2\xi = |\xi|\mathrm{d}|\xi|\mathrm{d}\theta$, we can express the expectation value of $\mathrm{e}^{\mathrm{i}\hat{\Theta}}$ in a normalized state $|\psi\rangle$ as

$$\begin{aligned} \langle\psi|\mathrm{e}^{\mathrm{i}\hat{\Theta}}|\psi\rangle &= \int \frac{\mathrm{d}^2\xi}{\pi}\ \langle\psi|\mathrm{e}^{\mathrm{i}\theta}|\xi\rangle_k\ {}_k\langle\xi|\psi\rangle \\ &= \int_0^{2\pi}\mathrm{d}\theta\mathrm{e}^{\mathrm{i}\theta}\int_0^\infty \frac{|\xi|\mathrm{d}|\xi|}{\pi}|\langle\xi|\psi\rangle|^2. \end{aligned} \tag{19}$$

Let $P(\theta) = \int_0^\infty \frac{|\xi|d|\xi|}{\pi}|\langle\xi|\psi\rangle|^2$, we have

$$\langle\psi|\mathrm{e}^{\mathrm{i}\hat{\Theta}}|\psi\rangle = \int_0^{2\pi}\mathrm{d}\theta\mathrm{e}^{\mathrm{i}\theta}P(\theta). \tag{20}$$

According to one of the postulates of the quantum mechanics [13]: "When the physical quantity A, to which orthonormalized eigenvectors $|u_n\rangle$ associated with the eigenvalue ω_n correspond, the expectation value of A in $|\psi\rangle$ is given by $\langle\psi|A|\psi\rangle = \Sigma_n|C_n|^2\omega_n$, where $|C_n|^2 = |\langle u_n|\psi\rangle|^2$ is the probability. "We have reason to name $P(\theta)$ in equation (20) the probability distribution function describing the degree of circular polarization, since $\mathrm{e}^{\mathrm{i}\theta}$ is the eigenvalue of $\mathrm{e}^{\mathrm{i}\hat{\Theta}}$. This shows that $|\xi\rangle_k$ spans a spectral representation of the operator $\sqrt{F/F^\dagger}$. Moreover, by introducing the number difference between left-handed and right-handed polarization operator $Q = a^\dagger a - b^\dagger b$, we see

$$[Q,F] = -F, \qquad [Q,F^\dagger] = F^\dagger. \tag{21}$$

Note that $\mathrm{e}^{\mathrm{i}\hat{\Theta}}$ is unitary,we can conclude:

$$[Q,\mathrm{e}^{\mathrm{i}\hat{\Theta}}] = -\mathrm{e}^{\mathrm{i}\hat{\Theta}}, \qquad [Q,\mathrm{e}^{-\mathrm{i}\hat{\Theta}}] = \mathrm{e}^{-\mathrm{i}\hat{\Theta}}. \tag{22}$$

We can also define the Hermitian operator of the angle as

$$\hat{\Theta} = \frac{1}{\pi}\int d^2\xi|\xi\rangle_k\ {}_k\langle\xi|\theta, \tag{23}$$

then as a result of equation (21), we have

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\hat{\Theta}}Q\mathrm{e}^{-\mathrm{i}\hat{\Theta}} = Q+1 = Q + [\mathrm{i}\hat{\Theta},\ Q] + \frac{1}{2!}[\mathrm{i}\hat{\Theta},[\mathrm{i}\hat{\Theta},Q]] \\ + \frac{1}{3!}[\mathrm{i}\hat{\Theta},[\mathrm{i}\hat{\Theta},[\mathrm{i}\hat{\Theta},Q]]] + ... \end{aligned} \tag{24}$$

which implies formally $[Q,\hat{\Theta}] = \mathrm{i}$.

4 Comparison with NFM operationally defined phase operator

It is worth comparing and contrasting equation (17) with the NFM operational phase operator, which is based on an eight-port homodyne experiment. As measurement always involve the difference between two phases, and as an homodyne experiment usually yields the cosine or sine of phase difference between two quantum states, the state of the input modes are an arbitrary two-mode state — the signal state in modes 10 and 1, and a coherent state (a local oscillator) — the reference, NFM proposed the cosine phase operator (in the limit of a strong local oscillator) as [10,11,14]

$$\begin{aligned} C &= \frac{\widehat{X}}{\sqrt{\widehat{X}^2+\widehat{P}^2}} \\ &= \frac{\widehat{x}_1+\widehat{x}_{10}}{\sqrt{(\widehat{x}_1+\widehat{x}_{10})^2+(\widehat{p}_1-\widehat{p}_{10})^2}} \end{aligned} \tag{25}$$

where

$$\begin{aligned} &x_1 = \frac{1}{\sqrt{2}}(\mathtt{a}+\mathtt{a}^\dagger), \quad x_{10} = \frac{1}{\sqrt{2}}(\mathtt{b}+\mathtt{b}^\dagger), \\ &p_1 = \frac{1}{\sqrt{2}\mathrm{i}}(\mathtt{a}-\mathtt{a}^\dagger), \quad p_{10} = \frac{1}{\sqrt{2}\mathrm{i}}(\mathtt{b}-\mathtt{b}^\dagger). \\ &[\mathtt{a},\mathtt{a}^\dagger] = [\mathtt{b},\mathtt{b}^\dagger] = 1. \end{aligned} \tag{26}$$

Substituting equation (26) into (25) leads to

$$C = \frac{\hat{X}}{\sqrt{\hat{X}^2 + \hat{P}^2}} = \frac{\mathtt{a} + \mathtt{a}^\dagger + \mathtt{b} + \mathtt{b}^\dagger}{2\sqrt{\mathtt{a}\mathtt{a}^\dagger + \mathtt{b}\mathtt{b}^\dagger + \mathtt{a}\mathtt{b} + \mathtt{a}^\dagger\mathtt{b}^\dagger}}$$

$$= \frac{1}{2}\left[\frac{\mathtt{a}^\dagger + \mathtt{b}}{\sqrt{(\mathtt{a}^\dagger + \mathtt{b})(\mathtt{b}^\dagger + \mathtt{a})}} + \frac{\mathtt{a} + \mathtt{b}^\dagger}{\sqrt{(\mathtt{a}^\dagger + \mathtt{b})(\mathtt{b}^\dagger + \mathtt{a})}}\right]$$

$$= \frac{1}{2}\left[\sqrt{\frac{\mathtt{a}^\dagger + \mathtt{b}}{\mathtt{a} + \mathtt{b}^\dagger}} + \sqrt{\frac{\mathtt{a} + \mathtt{b}^\dagger}{\mathtt{a}^\dagger + \mathtt{b}}}\right]$$

$$= \frac{1}{2}(\mathrm{e}^{-\mathrm{i}\alpha} + \mathrm{e}^{\mathrm{i}\alpha}) = \cos\alpha, \qquad (27)$$

where $\mathrm{e}^{\mathrm{i}\alpha} = \sqrt{\frac{\mathtt{a}+\mathtt{b}^\dagger}{\mathtt{a}^\dagger+\mathtt{b}}}$ is comparable in form with the operator $\sqrt{\frac{F}{F^+}} = \sqrt{\frac{af+b^\dagger f^*}{a^\dagger f^* + bf}}$ for polarization states of a photon. However,it must emphasized that although our operator F resembles in form to the NFM phase operator (see Eq. (27)), their physical meaning are completely different, since a single photon cannot carry phase information.

In summary, we have discussed the intrinsic relation between the quantized Riemann-Silberstein electromagnetic complex vector **F** and the determination of the nature of the polarization of a single photon. We have constructed **F**'s orthonormal and complete eigenvector set. Based on this, the new description reflecting the relative ratio between left-handed and right-handed polarization states of a photon is established quantum mechanically. Although our formalism seems to be comparable to NFM operationally defined phase operator, their physical conceptions are by no means the same, the operator (17) has absolutely nothing to do with optical phase in the usual sense, that is as the quantity shifted by a phase-shifter in a interferometer.

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